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20. ABSTRACT (Continue on reverse side if necessary and identify by block number)

Fisher Consistency of AM-Estimates of the Autoregression Parameter Using Hard Rejection Filter Cleaners

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ABSTRACT

An AM estimate $\hat{\phi}$ of the AR(1) parameter ϕ is a solution of the M-estimate equation $\sum_{1}^{n} \hat{x}_{t-1} \psi([y_t - \hat{\phi} \hat{x}_{t-1}]/s_t) = 0$ where $\hat{x}_{t-1}$,

$t = 0, 2, \ldots,$ satisfies the robust filter recursion

$\hat{x}_t = \hat{\phi} \hat{x}_{t-1} + s_t \psi^*([y_t - \hat{\phi} \hat{x}_{t-1}]/s_t)$, and s_t is a data dependent scale

which satisfies an auxiliary recursion. The AM-estimate may·be viewed as a

special kind of bounded-influence regression which provides robustness

toward contamination models of the type $y_t = (1 - z_t)x_t + z_t w_t$ where z_t

is a 0–1 process, w_t is a contamination process and x_t is an AR(1) process

with parameter ϕ. While AM-estimates have considerable heuristic appeal,

and cope with time series outliers quite well, they are not in general Fisher

consistent. In this paper, we show that under mild conditions, $\hat{\phi}$ is Fisher

consistent when ψ^* is of hard-rejection type.

February 4, 1987

1. INTRODUCTION

In recent years several classes of robust estimates of ARMA model parameters have been proposed. The three major classes of such estimates are: (i) GM-estimates (Denby and Martin, 1979; Martin, 1980; Bustos, 1982, Kunsch, 1984), (ii) AM-estimates (Martin, 1980; Martin, Samarov and Vandaele, 1983), and (iii) RA-estimates (Bustos, Fraiman and Yohai, 1984; Bustos and Yohai, 1986). See Martin and Yohai (1985) for an overview.

Each of the three types of estimates appear to have advantages over the others in certain circumstances. However, in some overall sense the AM-estimates seem most appealing: They are based in on an intuitively appealing robust filter-cleaner which "cleans" the data by replacing outliers with interpolates based on previous cleaned data. Furthermore, they have proved quite useful in a variety of applications (in addition to the references given after (ii) above, see also Kleiner, Martin and Thomson, 1979, and Martin and Thomson, 1982). On the other hand, the AM-estimates are sufficiently complicated functions of the data that it has proven difficult to establish even the most basic asymptotic properties such as consistency. Indeed, it appears that in general AM-estimates are not consistent (see the complaint of Anderson, 1983, in his discussion of Martin, Samarov and Vandaele, 1983), even though their asymptotic bias appears to be quite small (see the approximate bias calculation in Martin and Thomson, 1982).

In this paper we consider only a special case of AM-estimates based on a so-called *hard-rejection* filter cleaner. The importance of hard-rejection filter-cleaners, which are described in Section 2 for the first-order autoregressive (AR(1)) model, is that engineers often use a similar intuitively appealing modification of the Kalman filter for dealing with outliers in tracking problems. In Section 3 we prove that (under certain assumptions) these special AM-estimates are Fisher consistent for the parameter ϕ_0 of an AR(1) model, Fisher consistency being the first property one usually establishes along the way to proving consistency. In addition we prove uniqueness of the root of the asymptotic estimating

The AR(1) model we consider is

$$x_t = \phi_0 x_{t-1} + u_t, \quad t = 0, \pm1, \pm2, \cdots \quad (1.1)$$

along with the assumption

(A1)   *The u_t's are independent and identically distributed with symmetric distribution F which assigns positive probability to every interval.*

Furthermore, we shall let σ denote a measure of scale for the u_t's. For example, σ might be the median absolute deviation (MAD) of the u_t, scaled to yield the usual standard deviation when the u_t are Gaussian, namely, $\sigma = MAD/.6795$.

A model-oriented justification for using a robust procedure such as the AM-estimates treated here is that the observations are presumed to be given by the general contamination model

$$y_t = (1 - z_t^{\gamma}) x_t + z_t^{\gamma} w_t \quad (1.2)$$

where z_t^{γ} is a 0–1 process with $P(z_t^{\gamma} = 1) = \gamma + o(\gamma)$, and w_t is an outlier generating process. The processes z_t^{γ}, w_t and x_t are presumed jointly stationary. See, for example, Martin and Yohai (1986).

The filter-cleaners and AM-estimates introduced in the next section are designed to cope well with outliers generated by such a model. However, in this paper our main focus will be on the behavior of the AM-estimates only at the nominal model (1.1), i.e., when $z_t^{\gamma} \equiv 0$ in (1.2).

2. AM-ESTIMATES AND HARD-REJECTION FILTER CLEANERS

2.1 Filter Cleaners and AM-Estimates for the AR(1) Parameter

Suppose that the model (1.2) holds, for the moment with or without the condition $z_t^\gamma \equiv 0$.

Let $\hat{x}_t = \hat{x}_t(\phi)$ denote the filter-cleaner values generated for $t = 1, 2, \cdots$ by the robust filter cleaner recursion

$$\hat{x}_t = \phi \hat{x}_{t-1} + s_t \psi^* \left[\frac{y_t - \phi \hat{x}_{t-1}}{s_t} \right]$$

$$s_t^2 = \phi^2 p_{t-1} + \sigma^2 \tag{2.1}$$

$$p_t = s_t^2 \left[1 - w^* \left[\frac{y_t - \phi \hat{x}_{t-1}}{s_t} \right] \right]$$

with initial conditions

$$\hat{x}_0 = 0$$

$$s_0^2 = \frac{\sigma^2}{1 - \phi^2} . \tag{2.1'}$$

The robustifying psi-function ψ^* is odd and bounded, and the weight function w^* is defined by

$$w^*(r) = \frac{\psi^*(r)}{r} . \tag{2.2}$$

We shall often use the notation, $\hat{x}_t(\phi)$ and $s_t(\phi)$ to emphasize the dependence of $\hat{x}_t$ and s_t on ϕ. Then an AM-estimate $\hat{\phi}$ of ϕ is defined by

$$\sum_{t=2}^n \hat{x}_{t-1}(\hat{\phi}) \, \psi \left[\frac{y_t - \hat{\phi} \hat{x}_{t-1}(\hat{\phi})}{s_t(\hat{\phi})} \right] = 0 \tag{2.3}$$

where the robustifying function ψ is odd and bounded, but in general different than ψ^*. Since bounded ψ^* gives rise to bounded $\hat{x}_t$'s (see Martin and Su, 1986), the AM-estimate $\hat{\phi}$ can be regarded as a form of bounded influence regression (see Hampel et al., 1986). Let $\hat{\phi}_M$ be the "ordinary" M-estimate defined by

$$\sum_{t=2}^{n} y_{t-1} \psi \left[\frac{y_t - \hat{\phi}_M y_{t-1}}{s} \right] = 0 \qquad (2.4)$$

where s is some robust estimate of scale of the residuals $y_t - \hat{\phi}_M y_{t-1}$. The estimate $\hat{\phi}_M$ does not have bounded influence (see Martin and Yohai, 1986). The bounded influence estimate $\hat{\phi}$ efined by (2.3) is obtained from (2.4) by replacing y_{t-1} by $\hat{x}_{t-1}(\hat{\phi})$, and by replacing the global scale estimate s by the local, data-dependent scale s_t. Although the M-estimate $\hat{\phi}_M$ has high efficiency robustness at perfectly observed autoregressions (Martin, 1979), $\hat{\phi}_M$ is known to lack qualitative robustness (see for example Martin and Yohai, 1985), and the $\hat{\phi}$ of (2.3) represents a natural kind of robustification of $\hat{\phi}_M$.

We can characterize the asymptotic value of $\hat{\phi}$ as follows. First, assume that the filter recursions (2.1) are started not at $t = 0$, but in the remote past, and that $\hat{x}_t$, s_t and y_t are jointly asymptotically stationary. Then consider the equation

$$E \hat{x}_{t-1}(\phi(\mu)) \psi \left[\frac{y_t - \phi(\mu)\hat{x}_{t-1}(\phi(\mu))}{s_t(\phi(\mu))} \right] = 0 \qquad (2.5)$$

where μ is the measure for the process y_t, and the choice of t is arbitrary by virtue of starting the filter in the remote part. It is presumed that the functional $\phi(\mu)$ is well-defined by (2.5). Under reasonable conditions one expects that $\hat{\phi}$ is strongly or weakly consistent, i.e., that will converge to $\phi(\mu)$ almost surely, or in probability.

2.2 Fisher Consistency

A minimal requirement for any estimate, including robust estimates, is that of Fisher consistency. In the present context this means: when $z_t^\gamma \equiv 0$ in the general contamination model (1.2), we have $y_t \equiv x_t$ and then x_t has measure μ_{ϕ_0} where ϕ_0 is the true parameter value. Then $\hat\phi$ is said to be *Fisher consistent* if

$$\phi(\mu_{\phi_0}) = \phi_0 \qquad \forall \; \phi_0 \in (-1, 1). \tag{2.6}$$

In general, AM-estimates are not Fisher consistent. The plausibility of the claim is easy to see in the case where $\psi^* = \psi$. Substituting the basic filter equation of (2.1) in (2.3) gives:

$$\sum_{t=2}^{n} \hat x_{t-1}(\hat\phi) \frac{[\hat x_t(\hat\phi) - \hat\phi \hat x_{t-1}(\hat\phi)]}{s_t} = 0. \tag{2.7}$$

Thus, in this special case, $\hat\phi$ can be characterized as a weighted least squares estimate based on the cleaned data $\hat x_t = \hat x_t(\hat\phi)$. When $y_t \equiv x_t$ is an outlier free *Gaussian* process, a properly tuned filter-cleaner will result in $\hat x_t = x_t$ for most, *but not all,* times t. At those times t for which $\hat x_t \neq x_t$, $\hat x_t$ will typically be more highly correlated with $x_{t-1}, x_{t-2}, \cdots$, than is x_t. Thus, neither weighted nor classical least squares applied to the $\hat x_t$ is expected to yield consistent, or even Fisher consistent, estimates. This will be the case a fortiori when $y_t \equiv x_t$, but x_t has innovations outliers by virtue of the distribution of u_t having a heavy-tailed distribution (in which case the event $\hat x_t \neq x_t$ will occur more frequently).

The surprising result is that use of a hard-rejection filter cleaner does yield Fisher consistency under reasonable assumptions. In particular, according to our working assumption A1, the x_t process need not be Gaussian.

2.3 Hard-Rejection Filter Cleaners

From now on we take $z_t^\gamma \equiv 0$, and take ψ^* to be of the hard rejection type

$$\psi^*(r) = \begin{cases} r & |r| \le c \\ 0 & |r| > c \ . \end{cases} \tag{2.8}$$

Correspondingly

$$w^*(r) = \begin{cases} 1 & |r| \le c \\ 0 & |r| > c \ . \end{cases} \tag{2.10}$$

The constant c is adjusted to achieve a proper tradeoff between efficiency and robustness of the filter-cleaner (see Martin and Su, 1986, for guidelines here). The results in the remainder of the paper hold for any $c > 0$, and without lost of generality we take $c = 1$.

Note that when ψ^* in (2.1) is the hard-rejection type, the filter-cleaner value at time t is either $\hat{x}_t = y_t$ or $\hat{x}_t = \phi\hat{x}_{t-1}(\phi)$.

We can now characterize the hard-rejection filter as follows. Let the filter parameter be ϕ, and *from now on* replace y_t by x_t in (2.1). Then since $\psi^*(r)$ is either 0 or r in accordance with whether or not $|x_t - \phi\hat{x}_{t-1}(\phi)| \gtrless s_t$, it is easy to see that $\hat{x}_t(\phi)$ must have the form

$$\hat{x}_t(\phi) = \phi^{L_t} x_{t-L_t} \tag{2.11}$$

where $L_t = L_t(\phi)$ is the random time which has elapsed since the *last* "good" x_m. A "good" x_m is one for which $|x_m - \phi\hat{x}_{m-1}(\phi)| < s_m$, and hence $\hat{x}_m(\phi) = x_m$.

Let

$$N_t(\phi) = \text{the latest time, less or equal to } t \text{, at which a good } x_t \text{ occurs.} \tag{2.12}$$

Then

$$L_t(\phi) = t - N_t(\phi) \ . \tag{2.13}$$

Note from (2.1) with $y_t = x_t$, that for a good x_t we have $p_t = 0$ and $s_{t+1}^2 = \sigma^2$. Let

$$K_j^* = (\sigma^2 \sum_{k=0}^{j} \phi^{2k})^{\frac{1}{2}}, \quad j = 0, 1, 2, \cdots . \tag{2.14}$$

Then $s_t^2 = (K_l^*)^2$ if and only if $L_{t-1}(\phi) = l$

Now set

$$u_t(\phi) = x_t - \phi \hat{x}_{t-1}(\phi) \tag{2.15}$$

and note that the event M_t^* that x_t is bad (i.e., x_t is not good) occurs if and only if $u_t(\phi)$ is "rejected", i.e., if $|u_t(\phi)|$ is larger than the appropriate K_j^*. The appropriate K_j^* is $K_{L_{t-1}(\phi)}^*$, and so we can write

$$M_t^* = [\,|u_t(\phi)| \geq K_{L_{t-1}(\phi)}^*\,] . \tag{2.16}$$

Note that

$$M_t^* = [\hat{x}_t(\phi) = \phi \hat{x}_{t-1}(\phi), \quad N_t(\phi) = N_{t-1}(\phi)]$$

and

$$(M_t^*)^c = [\hat{x}_t(\phi) = x_t, \quad N_t(\phi) = t\,] .$$

For any j we can use (1.1) to write

$$x_t = \phi_0^j x_{t-j} + \sum_{k=0}^{j-1} \phi_0^k u_{t-k} . \tag{2.17}$$

If we set $j = L_{t-1}$ and $\phi(\mu_{\phi_0}) = \phi_0$, then (2.11) and (2.17) give

$$x_t - \phi_0 \hat{x}_{t-1}(\phi_0) = x_t - \phi_0^{1+L_{t-1}} x_{t-1-L_{t-1}}$$

$$= \sum_{k=0}^{L_{t-1}} \phi_0^k u_{t-k} .$$

In this case, with $y_t = x_t$ and $(\phi(\mu)) = \phi_0$, the left-hand side of (2.5) becomes

$$E \, \phi_0^{L_{t-1}-1} x_{t-1-L_{t-1}} \, \psi \left[\frac{\sum_{k=0}^{L_{t-1}} \phi_0^k \, u_{t-k}}{K_{L_{t-1}}^*(\phi_0)} \right]. \qquad (2.18)$$

Now if L_{t-1} were replaced by a fixed value m, then the independence of $u_{t-m}, \cdots, u_t$ and x_{t-m-1}, along with the evenness assumption on the distribution of the u_t and oddness assumption for ψ, would result in the above expectation being zero. This would give part of what is required to establish Fisher consistency — the other part is to show that (2.18) is non-zero when ϕ_0 is replaced by $\phi \neq \phi_0$. However, even for this first part a more detailed argument is required because $x_{t-L_{t-1}}$ and $u_{t-L_{t-1}}, \cdots, u_t$ are *not* conditionally independent, given $L_{t-1} = m$. Fortunately, symmetry and skewness arguments presented in the next section allow one to get around this difficulty.

3. THE FISHER CONSISTENCY RESULT

The following assumptions concerning ψ will be used.

(A2)   *The function* $\psi: \mathbf{R} \to \mathbf{R}$ *has the properties:*

   (i) ψ is monotone nondecreasing and odd

   (ii) ψ is strictly monotone on a neighborhood of zero.

   (iii) ψ is continuous

Definition: A distribution function F is called *right-skewed* (RS) if $F(x) + F(-x) \le 1$ for all x, and F is called *left-skewed* (LS) if $F(x) + F(-x) \ge 1$ for all x.

Proofs of Lemmas 1–4 below are elementary.

Lemma 1. Suppose that the random variable U has a distribution function F which gives positive probability to every neighborhood of the origin. Let ψ satisfy A2. If F is RS and $a > 0$, then $E\psi(a + U) > 0$. If F is LS and $a < 0$, then $E\psi(a + U) < 0$. If F is symmetric, then $E\psi(U) = 0$.

Lemma 2. Let X and Y be independent random variables, with the distribution of X being such that every interval has positive probability. Then the distribution of $X + Y$ gives positive probability to every interval.

Lemma 3. Let X and Y be independent random variables, with Y symmetric. If X is RS then so is $X + Y$, and if X is LS then so is $X + Y$.

Lemma 4. If U has a distribution F which is RS, then $\lambda > 0$ implies that the distribution of λU is RS and $\lambda < 0$ implies that it is LS.

The next two lemmas will also be used in order to establish Fisher consistency of $\phi(\mu)$.

Lemma 5. Let U have distribution F. For any constant $k > 0$ consider the event $M = [\,|a + U| \geq k\,]$, and let $F_{U|M}$ denote the distribution of U given M.

   (i)   If $a > 0$ and F is RS, then $F_{U|M}$ is RS.

   (ii) If $a < 0$ and F is LS, then $F_{U|M}$ is LS.

   (iii) If $a = 0$ and F is symmetric, then $F_{U|M}$ is symmetric.

Proof: The result (iii) is immediate, and since the arguments for (i) and (ii) are essentially the same we prove only (i). It suffices to show that for all $t \geq 0$ we have

$$P\,([U > t] \cap M) \;\geq\; P\,([U \leq -t] \cap M).\qquad(3.1)$$

Note that $M = [U \geq k - a] \cup [U \leq -k - a]$, and if $a > 0, t \geq 0$ we have

$$P([U \geq t] \cap M) \;=\; P\,(U \geq t,\; U \geq k - a)$$

and

$$P([U \leq -t] \cap M) \;=\; P\,(U \leq -t,\; U \geq k - a)$$

$$+ P\,(U \leq -t, U \leq -k - a).$$

These probabilities are readily compared for two separate cases.

Case a: $\qquad k - a \leq t, \ t \geq 0$

Here

$$P([U \geq t] \cap M) = P(U \geq t)$$

and

$$P([U \leq -t] \cap M) \leq P(U \leq -t)$$

Since $U \sim F$ with F RS, we get (3.1).

Case b: $\qquad 0 \leq t \leq k - a$

Now

$$P([U \geq t] \cap M) = P(U \geq k - a)$$

and

$$P([U \leq t] \cap M) = P(U \leq -k - a) \leq P(U \leq -(k-a))$$

which again gives (3.1). $\qquad \square$

Lemma 6: Let $U_1, U_2, \cdots$, be independent and identically distributed random variables with symmetric distribution function F. Let $a_1, a_2, \cdots$, and $h_2, h_3, \cdots$, be constants. Let $V_1 = U_1$ and for $i = 2, 3, \cdots$, let

$$V_i = h_i V_{i-1} + U_i. \tag{3.2}$$

Consider the events

$$M_i = [|a_i + V_i| \geq K_i], \quad i = 1, 2, \cdots$$

where K_1 is a constant, and for each $i \geq 2$ K_i is a function of $M_1, \cdots, M_{i-1}$. Set $M^n = \bigcap_{i=1}^{n} M_i$, and let $F_{V_n | M^n}$ be the conditional distribution of V_n given M^n.

(i) If $h_2 \geq 0, \ldots, h_n \geq 0$ and $a_1 \geq 0, \ldots, a_n \geq 0$, then $F_{V_n | M^n}$ is RS.

(ii) If $h_2 \geq 0, \ldots, h_n \geq 0$ and $a_1 \leq 0, \ldots, a_n \leq 0$, then $F_{V_n | M^n}$ is LS.

(iii) If $h_2 \leq 0, \ldots, h_n \leq 0$ and $a_1 \geq 0, a_2 \leq 0, \ldots, a_n(-1)^n \leq 0$, then $F_{V_n | M^n}$ is

RS or LS according if n is odd or even.

(iv) If $h_2 \leq 0, \ldots, h_n \leq 0$ and $a_1 \leq 0, a_2 \geq 0, \ldots, a_n(-1)^n \geq 0$, then $F_{V_n | M^n}$ is

LS or RS according if n is odd or even.

(v) If $a_1 = a_2 = \cdots = a_n = 0$, then $F_{V_n | M^n}$ is symmetric.

Proof: The proof is by induction. For $n = 1$,

$$M_1 = [\,|a_1 + U_1| \geq K_1\,]$$

and so (i)–(iii) follow from Lemma 5. Now suppose the result holds for $n - 1$, and consider the case (i). Then conditioned on M^{n-1}, $h_n V_{n-1}$ is RS and U_n is symmetric. From Lemma 3 it follows that conditioned on M^{n-1}, V_n is RS. Then since K_n is fixed, when we condition on M^{n-1}, use of Lemma 5 shows that $F_{V_n | M^n}$ is RS. A similar argument yields cases (ii) to (v). $\square$

Theorem: (Fisher Consistency) Suppose that F satisfies A1 and ψ satisfies A2. Furthermore, assume that the processes $\hat{x}_t$, s_t and x_t are jointly asymptotically stationary, and are governed by their asymptotic joint measure. If $\phi \phi_0 \geq 0$ and $\phi \neq \phi_0$ then for $t = 1, 2, \cdots,$

$$(\phi - \phi_0) E \, \hat{x}_{t-1}(\phi) \, \psi \left[\frac{r_t(\phi)}{s_t(\phi)} \right] < 0$$

where $r_t(\phi) = x_t - \phi \hat{x}_{t-1}(\phi)$, and

$$E \, \hat{x}_{t-1}(\phi_0) \, \psi \left[\frac{r_t(\phi_0)}{s_t(\phi)} \right] = 0 \, .$$

Proof: Let $\psi_r(u) = \psi \left[\dfrac{u}{K_r^*} \right]$, where K_r^* is given by (2.14), for any fixed $r \geq 0$,

consider the conditional expectation

$$E \left[\hat{x}_{t-1}(\phi) \, \psi \left[\frac{r_t(\phi)}{s_t(\phi)} \right] \mid N_{t-1}(\phi) = t-r-1, \, x_{t-r-1} \right]$$

$$= E \left[\hat{x}_{t-1}(\phi) \, \psi_r \, (r_t(\phi)) \mid N_{t-1}(\phi) = t-r-1, \quad x_{t-r-1} \right] .$$

Conditioned on $N_{t-1}(\phi) = t-r-1$ and x_{t-r-1} we have

$$\hat{x}_{t-r-1+i}(\phi) = \phi^i x_{t-r-1}, \qquad i = 0, 1, \cdots, r$$

and it follows from (1.1) that

$$x_{t-r-1+i} = \phi_0^i x_{t-r-1} + \sum_{l=0}^{i-1} \phi_0^l r_{t-r-1+i-l}, \quad i = 1, 2, \cdots, r+1 .$$

Thus, conditioned on $N_{t-1}(\phi) = t-r-1$, we have

$$r_{t-r-1+i}(\phi) = \sum_{l=0}^{i-1} \phi_0^l u_{t-r-1+i-l} + (\phi_0^i - \phi^i) x_{t-r-1}, \quad i = 1, 2, \cdots, r+1 .$$

Put

$$h_i \equiv \phi_0$$

$$a_i = (\phi_0^i - \phi^i) x_{t-r-1} \qquad i = 1, 2, \cdots, r+1$$

$$U_i = u_{t-r-1+i} \qquad\qquad i = 1, 2, \cdots, r+1 .$$

Let V_i, $1 \leq i \leq r$ be defined by (3.2) of in Lemma 6, so that

$$V_i = \sum_{l=0}^{i-1} \phi_0^l u_{t-r-1+i-l}, \quad i=1,2,\cdots,r+1.$$

and

$$r_{t-r-1+i}(\phi) = V_i + a_i, \quad i=1,2,\cdots,r+1.$$

Recalling the definition of M_t^* in (2.16), let

$$M_i = M_{t-r-1+i}^*, \quad i=1,2,\cdots,r+1$$

and note that conditioned on $N_{t-1}(\phi) = t-r-1$ and x_{t-r-1}, we are ready to apply Lemma 6 with $n = r+1$. We have

$$E\left[\hat{x}_{t-1}(\phi)\psi_r(r_t(\phi)) \mid N_{t-1}(\phi) = t-r-1, x_{t-r-1}\right]$$

$$= \phi^r x_{t-r-1} E \psi_r(V_{r+1}+a_{r+1} \mid M^r, x_{t-r-1}). \tag{3.3}$$

If $\phi = \phi_0$, then $a_1 = a_2 = \cdots = a_{r+1} = 0$, part (v) of Lemma 6 gives that $F_{V_r \mid M^r}$ is symmetric, and it follows from (3.2) that $F_{V_{r+1} \mid M^r}$ is symmetric as well. Then (3.3) is zero by Lemma 1.

Suppose first that $\phi_0 \in (0,1)$. If $0 < \phi < \phi_0$ and $x_{t-r-1} > 0$, then all the a_i's are positive and $F_{V_r \mid M^r}$ is RS by Lemma 6–(i). Then $F_{V_{r+1} \mid M^r}$ is RS by Lemma 3, and Lemmas 1–2, along with A1–A2, show that (3.3) is positive. Similarly, if $\phi < \phi_0$ and $x_{t-r-1} < 0$ then the a_i are all negative, $F_{V_r \mid M^r}$ and $F_{V_{r+1} \mid M^r}$ are both LS, which gives $E\left[\psi_r(V_{r+1}+a_{r+1}) \mid M^r\right] < 0$, and (3.3) is once again positive. Since $P(x_{t-r-1} = 0) = 0$, the result follows for $\phi \in (0,1), 0 < \phi < \phi_0$. A similar argument shows that (3.3) is negative for $\phi > \phi_0$.

Now suppose that $\phi_0 \in (-1,0)$. If $\phi < \phi_0 < 0$, $x_{t-r-1} > 0$ and r is odd, then we have $h_2 < 0, \ldots, h_r < 0$, $a_1 > 0$, $a_2 < 0, \ldots, a_r > 0$, $a_{r+1} < 0$. It follows from Lemma 6(iii) that $F_{V_r \mid M^r}$ is RS, and then by Lemmas 3–4 $F_{V_{r+1} \mid M^r}$ is LS. Hence

Lemmas 1–2 and A1–A2 yield $E\left[\psi_r(V_{r+1}+a_{r+1})\mid M^r\right]<0$. Since $\phi^r x_{t-r-n}<0$, (3.3) is positive. Similar arguments show that (3.3) is positive for r even, and also for $x_{t-r-n}<0$, r even or odd. Thus $E\left[\hat{x}_{t-1}(\phi)\,\psi(r_t(\phi))\mid M^r\right]<0$, for $\phi<\phi_0<0$. Similar arguments show that (3.3) is negative for $\phi_0<\phi<0$.

If $\phi_0=0$, then the above arguments reveal that (3.3) is positive for $\phi<0$ and negative for $\phi>0$.

The result follows by averaging over the conditioning in (3.3). $\qquad\square$

4. CONCLUDING REMARKS

The theorem in Section 3 does not in fact give uniqueness of the root of (2.5) unless we know the sign of ϕ_0. At the present time, we have good reason to believe that the inequality of the theorem does not hold for all $\phi \in (-1, 1)$. However, in the case that (.25) has a root may sign, we still can be Fisher consistent by choosing as estimate the root minimizing

$$\sum_{t=2}^{n} \left[\frac{\hat{x}_t - \phi \hat{x}_{t-1}}{s_t} \right]^2 .$$

It would be nice to obtain Fisher consistency for the AR(p) case. Unfortunately, Fisher consistency does not hold for the $p\,th$ order analogue ($p \geq 2$) of the hard-rejection filter-based AM-estimated treated here. It appears, however, that one or more modifications may yield Fisher consistency.

These questions will be pursued elsewhere.

References

Bustos, O.H. (1982). "General M-estimates for contaminated p th-order autoregressive processes: consistency and asymptotic normality." *Z. Wahrsch.* **59**, 491–504.

Bustos, O.H., Fraiman, R. and Yohai, V.J. (1984). "Asymptotic behavior of the estimates based on residual autocovariances for ARMA models." In *Robust and Nonlinear Times Series Analysis,* edited by J. Franke, W. Härdle, and D. Martin, pp. 26–49. Springer, New York.

Bustos, O. and Yohai, V.J. (1986). "Robust estimates for ARMA models." *Journal of the American Statistical Association,* **81**, 155–168.

Denby, L. and Martin, R.D. (1979). "Robust estimation of the first order autoregressive parameter." *Jour. Amer. Stat. Assoc.* **74**, 140–146.

Hampel, F.R., Ronchetti, E.M., Rousseeuw, P.J. and Stahel, W.A. (1986). *Robust Statistics: The Approach Based on Influence Functions.* Wiley, New York.

Kleiner, B., Martin, R.D. and Thompson, D.J. (1979). "Robust estimation of power spectra." *Jour. Roy. Stat. Soc., Series B,* **41**, 313–351.

Künsch, H. (1984). "Infinitesimal robustness for autoregressive processes." *The Annals of Statistics,* **12**, 843–863.

Mallows, C.L. (1980). "Some theory of nonlinear smoothers." *The Annals of Statistics* **8**, 695–715.

Martin, R.D. (1980). "Robust estimation of autoregressive models (with discussion)." In *Directions in Time Series,* edited by D.R. Brillinger and G.C. Tiao. Instit. of Math. Statistics Publication, Haywood, CA, pp. 228–254.

Martin, R.D. (1981). "Robust methods for time series." In *Applied Time Series II,* edited by D.F. Findley. Academic Press, New York, pp. 683–759.

Martin, R.D. (1982). "The Cramer-Rao bound and robust M-estimates for autoregressions." *Biometrika* **69**, 437–442.

Martin, R.D., Samarov, A. and Vandaele, W. (1983). "Robust methods for ARIMA models." In *Applied Time Series Analysis of Economic Data*, edited by A. Zellner. Econ. Res. Report ER-t, Bureau of the Census, Washington, DC.

Martin, R.D. and Thompson, D.J. (1982). "Robust resistant spectrum estimation." *IEEE Proceedings*, Vol. **70**, 1097–1115.

Martin, R.D. and Yohai, V.J. (1985). "Robustness in time series and estimating ARMA models." In *Handbook of Statistics*, **5**, edited by E.J. Hannan, P.R. Krishnaiah and M.M. Rao, pp. 119–155. Elsevier, New York.

Martin, R.D. and Yohai, V.J. (1986). "Influence functionals for time series." *The Annals of Statistics*, **14**, 781–855.

Papantoni-Kazakos, P. and Gray, R.M. (1979). "Robustness of estimators on stationary observations." *Ann. Probab.* **7**, 989–1002.